Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
International Advanced Level

Centre Number | Candidate Number

Wednesday 22 January 2020

Morning (Time: 1 hour 30 minutes) | Paper Reference **WMA13/01**

Mathematics

International Advanced Level
Pure Mathematics P3

You must have:
Mathematical Formulae and Statistical Tables (Lilac), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

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P 6 0 5 6 8 A 0 1 3 2

Pearson

1. A population of a rare species of toad is being studied.

The number of toads, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{900e^{0.12t}}{2e^{0.12t} + 1} \qquad t \geqslant 0, t \in \mathbb{R}$$

According to this model,

(a) calculate the number of toads in the population at the start of the study,

(1)

(b) find the value of t when there are 420 toads in the population, giving your answer to 2 decimal places.

(4)

(c) Explain why, according to this model, the number of toads in the population can never reach 500

(1)

1. a) at the start of the study $t=0$

$$\therefore N_0 = \frac{900e^{0.12(0)}}{2e^{0.12(0)}+1} = \frac{900}{2+1} = 300 \text{ toads}$$

b) $$420 = N_t = \frac{900e^{0.12t}}{2e^{0.12t}+1}$$

$$420(2e^{0.12t}+1) = 900e^{0.12t}$$

$$60e^{0.12t} = 420$$

$$e^{0.12t} = 7$$ ← take natural logarithms of both sides

$$\ln(e^{0.12t}) = \ln(7)$$

$\ln(e^{0.12t})$
$= 0.12t(\ln(e))$
$= 0.12t$

$$0.12t = \ln(7)$$

$$\therefore t = \frac{\ln(7)}{0.12} = 16.22 \text{ years}$$

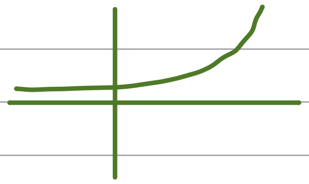
P 6 0 5 6 8 A 0 2 3 2

Question 1 continued

c) $N = \dfrac{900e^{0.12t}}{2e^{0.12t}+1}$ ← divide numerator & denominator by $e^{0.12t}$

$= \dfrac{900}{2+\dfrac{1}{e^{0.12t}}}$

as $t \to \infty$ $\quad e^{0.12t} \to \infty$ $\quad \therefore \dfrac{1}{e^{0.12t}} \to 0$



$\therefore$ as $t \to \infty$

$N_{\text{max limit}} = \dfrac{900}{2+\dfrac{1}{e^{0.12t}}} \to \dfrac{900}{2+0} = 450$

$\therefore$ If the upper limit to the population of toads is 450 it can never reach 500

Q1

(Total 6 marks)

P 6 0 5 6 8 A 0 3 3 2

Turn over

2. The function f and the function g are defined by

$$f(x) = \frac{12}{x+1} \qquad x > 0, x \in \mathbb{R}$$

$$g(x) = \frac{5}{2}\ln x \qquad x > 0, x \in \mathbb{R}$$

(a) Find, in simplest form, the value of $fg(e^2)$ **(2)**

(b) Find f^{-1} **(3)**

(c) Hence, or otherwise, find all real solutions of the equation

$$f^{-1}(x) = f(x)$$

(3)

2. a) $f(x) = \frac{12}{x+1}$ $g(x) = \frac{5}{2}\ln(x)$

$f(g(e^2)) = f\left(\frac{5}{2}\ln(e^2)\right)$

$= f(5)$

$= \frac{12}{5+1} = \frac{12}{6} = 2$

b) to find $f^{-1}(x)$: $f(x) = \frac{12}{x+1}$

① write the function using a "y" and set equal to "x" : $x = \frac{12}{y+1}$

② rearrange to make y the subject : $y+1 = \frac{12}{x}$

$y = \frac{12}{x} - 1$

③ replace y with $f^{-1}(x)$: $f^{-1}(x) = \frac{12}{x} - 1$

Because we are told to find $f^{-1}(x)$, we must also state the domain of the inverse function :

↑ domain refers to the set of values we are allowed to plug into our function

domain of inverse function = range of function

↑ range refers to all possible values of a function

$\therefore$ domain of $f^{-1}(x)$ = range of $f(x)$

$f(x) = \frac{12}{x+1}$ $x > 0, x \in \mathbb{R}$

$\therefore$ range of $f(x)$ is $0 < f(x) < 12$

$\therefore f^{-1}(x) = \frac{12}{x} - 1$ $0 < x < 12$

c) $f^{-1}(x) = f(x)$

$\frac{12}{x+1} = \frac{12}{x} - 1$

$12x = 12(x+1) - x(x+1)$

$\cancel{12x} = \cancel{12x} + 12 - x^2 - x$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

must reject $x = -4$ because $0 < x < 12$ & -4 is not in domain

$\therefore x = \cancel{-4} \cup 3$

$\therefore x = 3$

Q2

(Total 8 marks)

3.

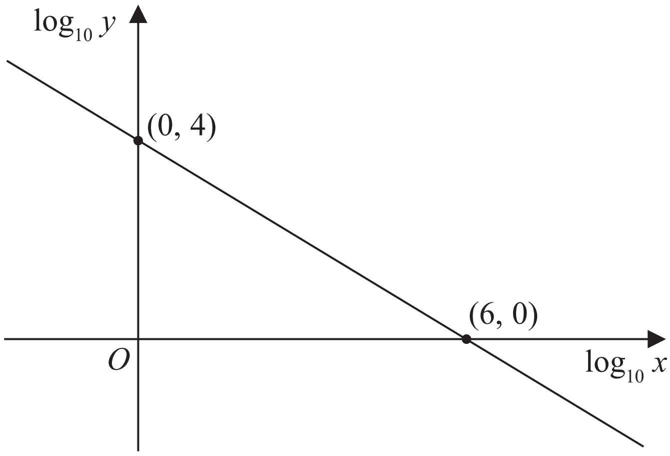


Figure 1

Figure 1 shows a linear relationship between $\log_{10} y$ and $\log_{10} x$

The line passes through the points (0, 4) and (6, 0) as shown.

(a) Find an equation linking $\log_{10} y$ with $\log_{10} x$
(2)

(b) Hence, or otherwise, express y in the form px^q, where p and q are constants to be found.
(3)

3. a) Equation of line : $y - y_1 = m(x - x_1)$

coordinates of known point on line

gradient

Known points on line : $(0,4)$
$(6,0)$

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{6-0} = -\frac{2}{3}$$

$$\log_{10} y - 0 = -\frac{2}{3}(\log_{10} x - 6)$$

$$\log_{10} y = -\frac{2}{3}\log_{10} x + 4$$

b) LOG RULES →

$$a \log_b(c) = \log_b(c^a)$$

$$\log_a b + \log_a c = \log_a(bc)$$

$$\log_a b = c \rightarrow a^c = b$$

Question 3 continued

$\log_{10} y = -\log_{10}(x^{\frac{2}{3}}) + 4$ ← using 1st log rule

$\log_{10} y + \log_{10}(x^{\frac{2}{3}}) = 4$

$\log_{10}(yx^{2/3}) = 4$ ← using 2nd log rule

$yx^{2/3} = 10^4$ ← using 3rd log rule

$y = 10000x^{-\frac{2}{3}}$

$p = 10000 \qquad q = -\frac{2}{3}$

Q3

(Total 5 marks)

4. (i)

$$f(x) = \frac{(2x+5)^2}{x-3} \qquad x \neq 3$$

(a) Find $f'(x)$ in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are fully factorised quadratic expressions.

(b) Hence find the range of values of x for which $f(x)$ is increasing.

(6)

(ii)

$$g(x) = x\sqrt{\sin 4x} \qquad 0 \leqslant x < \frac{\pi}{4}$$

The curve with equation $y = g(x)$ has a maximum at the point M.

Show that the x coordinate of M satisfies the equation

$$\tan 4x + kx = 0$$

where k is a constant to be found.

(5)

4. (i) $f(x) = \frac{(2x+5)^2}{x-3}$ $\quad x \neq 3$

Quotient rule for differentiating : $y = \frac{u}{v} \rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$u = (2x+5)^2$ $\qquad \frac{du}{dx} = 8x + 20$ $\quad \leftarrow$ differentiate after expanding

$= 4x^2 + 20x + 25$

OR

$u = (2x+5)^2$ $\qquad \frac{du}{dx} = \frac{du}{\cancel{dy}} \times \frac{\cancel{dy}}{dx}$ $\quad \leftarrow$ differentiate using chain rule

$u = (y)^2$

where $y = 2x+5$ $\qquad = 2y \times 2$

$\frac{du}{dy} = 2y$ $\qquad \frac{dy}{dx} = 2$ $\qquad = 4y = 4(2x+5)$

$= 8x + 20$

Question 4 continued

$v = x - 3 \qquad \frac{dv}{dx} = 1$

$$f'(x) = \frac{(x-3)(8x+20) - (1)((2x+5)^2)}{(x-3)^2}$$

$$= \frac{(2x+5)(4x-12-2x-5)}{(x-3)^2} = \frac{(2x+5)(2x-17)}{(x-3)^2}$$

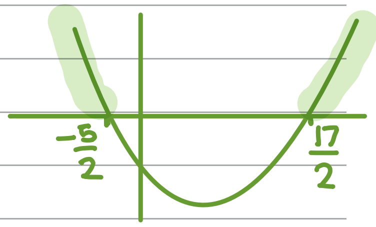
b) $f(x)$ is increasing when $f'(x) > 0$

$$\frac{(2x+5)(2x-17)}{\underbrace{(x-3)^2}} > 0$$

always positive as $(\text{real no.})^2$ is always ≥ 0

$\therefore$ for $f'(x) > 0$

$(2x+5)(2x-17) > 0 \rightarrow$



$x < -\frac{5}{2} \quad \cup \quad x > \frac{17}{2}$

(ii) $g(x) = x\sqrt{\sin(4x)} \qquad 0 \leq x < \frac{\pi}{4}$

At maxima & minima, $g'(x) = 0$

$\boxed{\text{PRODUCT RULE : } \quad y = uv \quad y' = u'v + uv'}$

$g(x) = x \times (\sin(4x))^{\frac{1}{2}}$

Let $u = x \rightarrow \frac{du}{dx} = 1$

Question 4 continued

$v = \sqrt{\sin(4x)} = (\sin(4x))^{\frac{1}{2}}$

$$\boxed{\text{CHAIN RULE}: \quad \frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}}$$

$v = u^{\frac{1}{2}} \qquad u = \sin(4x)$

$$\frac{dv}{dx} = \frac{dv}{\cancel{du}} \times \frac{\cancel{du}}{dx} = \frac{1}{2} \times u^{-\frac{1}{2}} \times 4\cos(4x)$$

$$= \frac{1}{2}(\sin(4x))^{-\frac{1}{2}} \times 4\cos(4x)$$

$$= \frac{2\cos(4x)}{\sqrt{\sin(4x)}}$$

$$g'(x) = (1)\left((\sin(4x))^{\frac{1}{2}}\right) + (x)\left(\frac{2\cos(4x)}{\sqrt{\sin(4x)}}\right)$$

$$= \sqrt{\sin(4x)} + x\frac{2\cos(4x)}{\sqrt{\sin(4x)}}$$

$$= \frac{\sin(4x) + 2x\cos(4x)}{\sqrt{\sin(4x)}}$$

At maxima : $g' = 0 \quad \therefore \dfrac{\sin(4x) + 2x\cos(4x)}{\sqrt{\sin(4x)}} = 0$

$\sin(4x) + 2x\cos(4x) = 0 \rightarrow \sin(4x) = -2x\cos(4x)$

$$\tan(4x) = -2x$$

$$\tan(4x) + 2x = 0$$

$$k = 2$$

5. (a) Use the substitution $t = \tan x$ to show that the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

can be written in the form

$$5t^4 - 24t^2 - 5 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 360°$, the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

Show each stage of your working and give your answers to one decimal place.

(4)

5. a) $12\tan(2x) + 5\cot(x)\sec^2(x) = 0$

$t = \tan x$

USING DOUBLE ANGLE IDENTITIES $\rightarrow$ $\boxed{\tan(2A) = \dfrac{2\tan(A)}{1-\tan^2(A)}}$

$\sin^2 A + \cos^2 A = 1$ $\leftarrow$ divide through by $\cos^2 A$

$\dfrac{\sin^2 A}{\cos^2 A} + \dfrac{\cos^2 A}{\cos^2 A} = \dfrac{1}{\cos^2 A}$

$\boxed{\tan^2 A + 1 = \sec^2 A}$

Using 2 identities from above

$12\left(\dfrac{2\tan(t)}{1-\tan^2(t)}\right) + 5 \times \dfrac{1}{\tan(t)} \times (\tan^2(t)+1) = 0$

$\dfrac{24t}{1-t^2} + \dfrac{5}{t}(t^2+1) = 0$

$24t^2 + 5(1+t^2)(1-t^2) = 0$

$5(1-t^4) + 24t^2 = 0$

P 6 0 5 6 8 A 0 1 4 3 2

Question 5 continued

$5t^4 - 24t^2 - 5 = 0$

b) $12\tan(2x) + 5\cot(x)\sec^2(x) = 0$

$5t^4 - 24t^2 - 5 = 0$

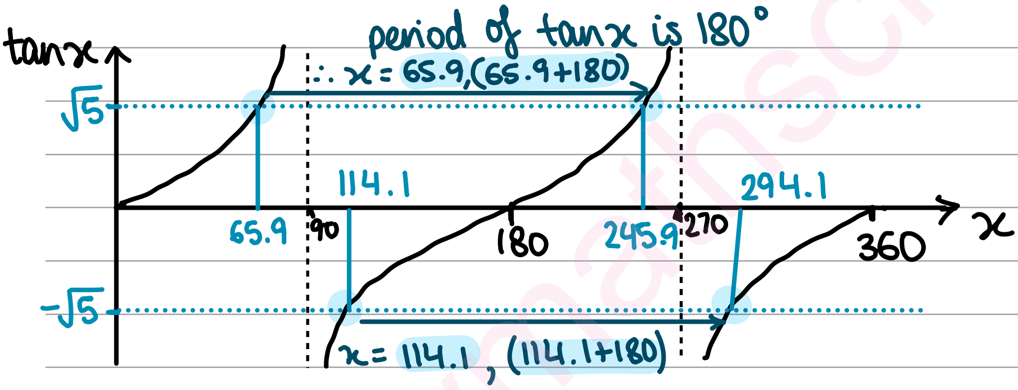
$(5t^2 + 1)(t^2 - 5) = 0$

$t^2 = \cancel{-\frac{1}{5}} \cup 5$

reject negative solution for t^2

$\therefore t^2 = 5$

$t = \pm\sqrt{5}$



$\therefore x = 65.9°, 114.1°, 245.9°, 294.1°$

6.

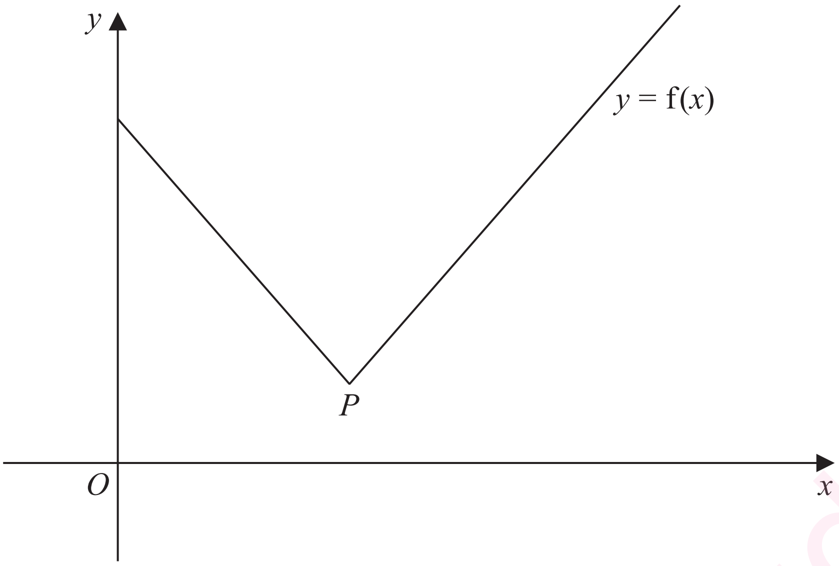


Figure 2

Figure 2 shows part of the graph with equation $y = \text{f}(x)$, where

$$\text{f}(x) = 2\,|\,2x - 5\,| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\text{f}(x) = 3x - 2$

(4)

Given that the equation

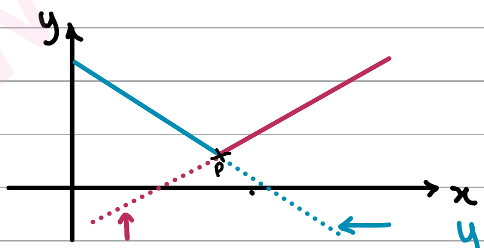
$$\text{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

6. a) $f(x) = 2|2x-5| + 3 \qquad x \geqslant 0$



$y = -2(2x-5) + 3$

$y = 2(2x-5) + 3$

Question 6 continued

P is the point at which the 2 lines meet

$$2(2x-5)\cancel{+3} = -2(2x-5)\cancel{+3}$$

$$4x - 10 = -4x + 10$$

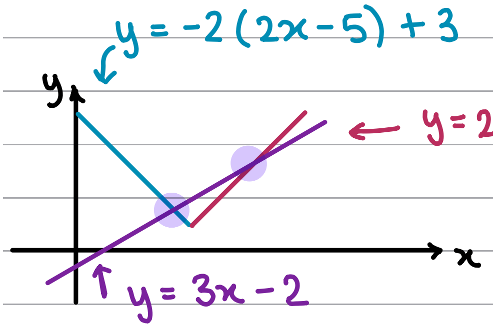
$$8x = 20$$

$$x = \frac{5}{2}$$

$\therefore$ coordinates of P are

$(2.5, 3)$

b) $f(x) = 3x - 2$



The line $y = 3x-2$ intersects with both parts of $f(x)$ $\therefore$ there are 2 solutions

$$-2(2x-5)+3 = 3x-2$$

$$-4x+10+3 = 3x-2$$

$$7x = 15$$

$$x = \frac{15}{7}$$

$$2(2x-5)+3 = 3x-2$$

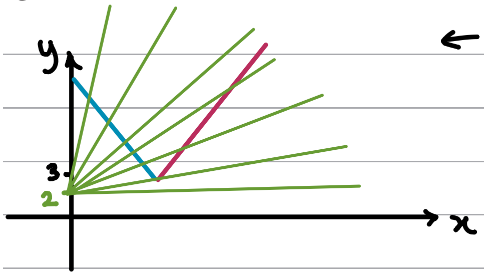
$$4x-10+3 = 3x-2$$

$$x = 5$$

c) given that $f(x) = kx+2$ has exactly 2 roots,

the line $y = kx+2$ must intersect with both parts of $f(x)$

Question 6 continued



← The green lines represent some possibilities of the line $y = kx + 2$

as k changes, the gradient changes (& so does the no. of intersections with $f(x)$)

gradient of $f(x) = -4$ & 4

To find the range of k, we must look at all possible scenarios with diff. values of k:

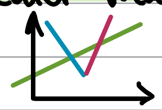
* when k is very low, there are 0 solutions as $y = kx + 2$ doesn't intersect with $f(x)$

e.g. 

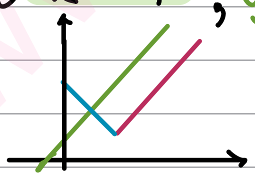
* when k is such that $y = kx + 2$ passes through exactly P, there is only 1 solution

at this point $k = \frac{3-2}{2.5-0} = \frac{2}{5}$

* when k is slightly greater than $\frac{2}{5}$, $y = kx + 2$ intersects with both parts of $f(x)$ ∴ 2 distinct solutions



* when $k = 4$, $y = kx + 2$ is parallel to part of $f(x)$



∴ there will only ever be an intersection with the blue part
∴ only 1 solution

∴ to have 2 distinct solutions

$$\frac{2}{5} < k < 4$$

7.

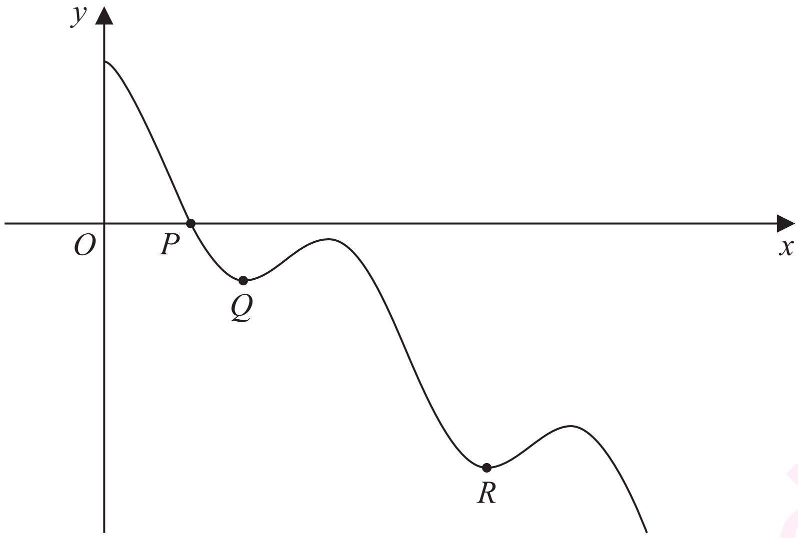


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

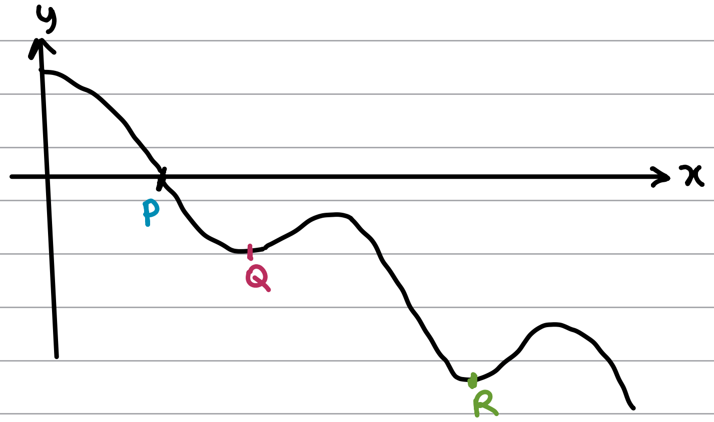
Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

Question 7 continued

7. a)



At P, $y = 0$

└ as we can see from the graph

just before P $\rightarrow$ $y > 0$

& just after P $\rightarrow$ $y < 0$

$y = 2\cos(3x) - 3x + 4$

$y_{0.8} = 2\cos(3\times 0.8) - 3(0.8) + 4 = 0.125\ldots$

$y_{0.9} = 2\cos(3\times 0.9) - 3(0.9) + 4 = -0.508\ldots$

} there has been a sign change

& because the graph is continuous between the specified points, the graph must cross the x-axis between 0.8 & 0.9

so P lies between 0.8 & 0.9

$\therefore\ 0.8 < \alpha < 0.9$

b)(i) $x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$

$x_1 = 0.8$

$x_{n+1} = x_{1+1} = x_2 = \frac{1}{3}\arccos(1.5(0.8) - 2)$

$= \frac{1}{3}\arccos(-0.8) = 0.8327$

Question 7 continued

$$x_3 = \frac{1}{3}\arccos\left(1.5x_2 - 2\right)$$

$$= \frac{1}{3}\arccos\left(1.5(0.8327) - 2\right) = 0.8068$$

$$x_4 = \frac{1}{3}\arccos\left(1.5x_3 - 2\right) = 0.8271$$

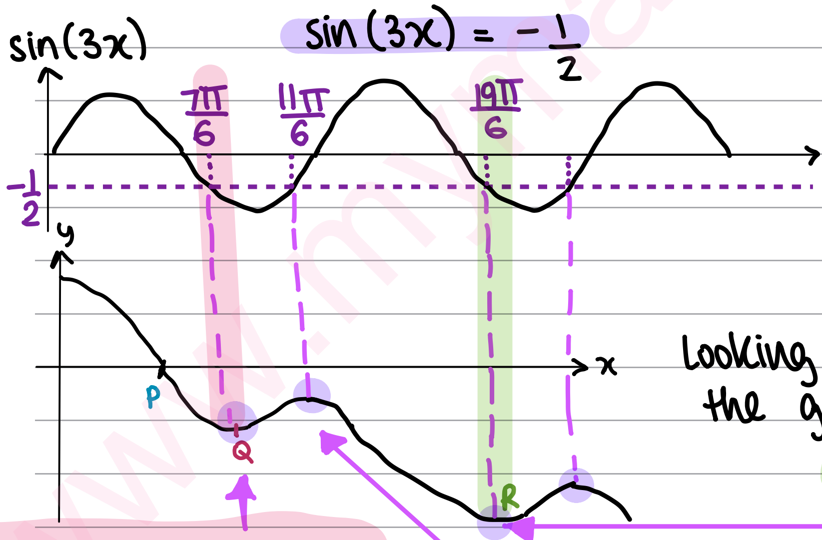
$$x_5 = 0.8110$$

c) At local minima & maxima, $\frac{dy}{dx} = 0$

$$y = 2\cos(3x) - 3x + 4$$

$$\frac{dy}{dx} = 2 \times -3\sin(3x) - 3$$

$$= -6\sin(3x) - 3 = 0$$



All points where $\frac{dy}{dx} = 0$ are critical points

(i.e. local maxima/minima)

Looking again at the graph:

the first stationary point is Q

$\therefore \beta$ is the 1st positive value of x for which $\sin(3x) = -\frac{1}{2}$

$\therefore \beta = \frac{7\pi}{6} \times \frac{1}{3} = \frac{7\pi}{18}$

the 2nd value for x would give the local maxima $\therefore x_{max} = \frac{11\pi}{18}$

The 3rd positive value gives R

$\therefore \gamma = \frac{1}{3} \times \frac{19\pi}{6}$

$= \frac{19\pi}{18}$

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,dx$$

giving your answer in simplest form.

(4)

(ii) $$h(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $h(x) = Ax + B + \dfrac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int h(x)\,dx$$

(6)

8.(i) $\displaystyle\int_3^{42} \frac{2}{3x-1}\,dx$

$= \dfrac{2}{3}\displaystyle\int_3^{42} \frac{1}{x-\frac{1}{3}}\,dx$

$= \dfrac{2}{3}\left[\ln\left(x-\frac{1}{3}\right)\right]_3^{42}$

$= \dfrac{2}{3}\left(\ln\left(\frac{125}{3}\right) - \ln\left(\frac{8}{3}\right)\right)$

$= \dfrac{2}{3}\ln\left(\frac{125}{8}\right) = \ln\left(\left(\frac{125}{8}\right)^{\frac{2}{3}}\right) = \ln\left(\frac{25}{4}\right)$

(ii) $h(x) = \dfrac{2x^3-7x^2+8x+1}{(x-1)^2} \qquad x>1$

$h(x) = Ax + B + \dfrac{C}{(x-1)^2}$

Question 8 continued

$$\begin{array}{r} 2x - 3 \\ (x^2-2x+1)\overline{)2x^3-7x^2+8x+1} \\ -\underline{2x^3-4x^2+2x} \\ -3x^2+6x+1 \\ -\underline{-3x^2+6x-3} \\ 4 \end{array}$$

$\leftarrow$ remainder

$\therefore (x^2-2x+1)(2x-3) + 4 = 2x^3-7x^2+8x+1$

$\therefore h(x) = 2x - 3 + \dfrac{4}{(x-1)^2}$

$A = 2$
$B = 3$
$C = 4$

$\rightarrow \int h(x)\,dx$

$= \int 2x - 3 + 4(x-1)^{-2}\,dx$

$= x^2 - 3x - 4(x-1)^{-1} + c$ $\Leftarrow$ remember constant of integration when there are no limits

$\therefore \int h(x)\,dx = x^2 - 3x - \dfrac{4}{x-1} + c$

9. $$f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$$

(a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = f(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$g(\theta) = \frac{90}{4 + (f(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

9. a) $f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in R$

$f(\theta) = R\cos(\theta + \alpha)$ ← using compound angle formulae

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$f(\theta) = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

compare expanded expression to $f(\theta)$ given

$R\cos\theta\cos\alpha - R\sin\theta\sin\alpha = 5\cos\theta - 4\sin\theta$

$R\cos\alpha = 5 \qquad R\sin\alpha = 4$

$\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \tan\alpha = \frac{4}{5}$

$\alpha = 0.675$

$(R\cos\alpha)^2 + (R\sin\alpha)^2$ — $\cos^2 A + \sin^2 A = 1$ identity

$= R^2(\cos^2\alpha + \sin^2\alpha) = R^2(1)$

$= 5^2 + 4^2$

$\therefore R^2 = 41 \qquad R = \cancel{\pm}\sqrt{41}$

given that $R > 0$

Question 9 continued

b)(i) $y = \cos(\theta)$

stretch

translation

$y = 5\cos(\theta) - 4\sin(\theta) = \sqrt{41}\cos(\theta + 0.675)$

stretch : $y = \cos(\theta) \longrightarrow y = \sqrt{41}\cos(\theta)$

stretch
scale factor $\sqrt{41}$
parallel to y-axis

(ii) translation : $y = \sqrt{41}\cos(\theta) \longrightarrow y = \sqrt{41}\cos(\theta + 0.675)$

translation
through the vector $\begin{pmatrix} -0.675 \\ 0 \end{pmatrix}$

c) $g(\theta) = \dfrac{90}{4 + (f(\theta))^2} \qquad \theta \in R$

Range → all possible values of $g(\theta)$

range of $f(\theta)$: $-\sqrt{41} \leq f(\theta) \leq \sqrt{41}$

$\therefore \quad 0 \leq (f(\theta))^2 \leq 41$

so upper limit of $g(\theta)$ → when $(f(\theta))^2 = 0$

$g(\theta) = \dfrac{90}{4+0} = \dfrac{45}{2}$

lower limit of $g(\theta)$ → when $(f(\theta))^2 = 41$

$g(\theta) = \dfrac{90}{4+41} = 2$

$\therefore \quad 2 \leq g(\theta) \leq \dfrac{45}{2}$

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